

ASYMPTOTICS OF CARLEMAN POLYNOMIALS FOR LEVEL CURVES OF THE INVERSE OF A SHIFTED ZHUKOVSKY TRANSFORMATION

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ABSTRACT. This paper complements the recent investigation of [4] on the asymptotic behavior of polynomials orthogonal over the interior of an analytic Jordan curve L . We study the specific case of $L = \{z = w - 1 + (w - 1)^{-1}, |w| = R\}$, for some $R > 2$, providing an example that exhibits the new features discovered in [4], and for which the asymptotic behavior of the orthogonal polynomials is established over the entire domain of orthogonality. Surprisingly, this variation of the classical example of the ellipse turns out to be quite sophisticated. After properly normalizing the corresponding orthonormal polynomials p_n , $n = 0, 1, \dots$, and on certain critical subregion of the orthogonality domain, a subsequence $\{p_{n_k}\}$ converges if and only if $\log_{\mu^4}(n_k)$ converges modulo 1 (μ being an important quantity associated to L). As a consequence, the limiting points of the sequence $\{p_n\}$ form a one parameter family of functions, the parameter's range being the interval $[0, 1)$. The polynomials p_n are much influenced by a certain integrand function, the explained behavior being the result of this integrand having a nonisolated singularity that is a cluster point of poles. The nature of this singularity sparks purely from geometric considerations, as opposed to the more common situation where the critical singularities come from the orthogonality weight.

1. INTRODUCTION AND NEW RESULTS

The study of orthogonal polynomials over planar regions seems to have originated in the work of T. Carleman [2], followed up by contributions from several authors, but more prominently by P. K. Suetin (see his monograph [17] and the many references therein).

Recently, the subject has experienced a new surge, with many new interesting results in a variety of topics such as the asymptotic behavior and zero distribution of the orthogonal polynomials [4, 6, 7, 8, 10, 11, 12, 14, 16], universality and Christoffel functions [9, 14, 20], and the existence of recurrence relations [1, 18, 19]. In particular, [6] and [20] consider orthogonality over several domains, while [14] considers orthogonality with respect to certain potential theoretic varying weights. The papers [13] and [15], though more general in scope, also discuss important implications for planar orthogonality.

Let G_1 be a bounded simply-connected domain of \mathbb{C} , whose boundary L_1 is an analytic Jordan curve, and let $\{p_n(z)\}_{n=0}^{\infty}$ be the unique sequence of polynomials

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satisfying that p_n is a polynomial of degree n with positive leading coefficient, and that

$$\frac{1}{\pi} \int_{G_1} p_n(z) \overline{p_m(z)} dx dy = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

These are the polynomials originally investigated by Carleman in [2]. Among other things, Carleman derived an asymptotic formula that establishes the behavior of $p_n(z)$ as $n \rightarrow \infty$ on certain neighborhood of $\overline{\mathbb{C}} \setminus G_1$ (the set denoted by Ω_ρ below). To state this result with precision, we first need to convene on some notation.

For each $r > 0$, we define

$$\mathbb{T}_r := \{w : |w| = r\}, \quad \Delta_r := \{w : r < |w| \leq \infty\}, \quad \mathbb{D}_r := \{w : |w| < r\}.$$

Let Ω_1 be the unbounded component of $\overline{\mathbb{C}} \setminus L_1$, and let $\psi(w)$ be the unique conformal map of Δ_1 onto Ω_1 that satisfies $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. Because L_1 is analytic, there is a smallest number $0 \leq \rho < 1$ for which ψ admits an analytic and univalent continuation to Δ_ρ , and we define

$$\phi(z) : \Omega_\rho \rightarrow \Delta_\rho$$

to be the inverse of ψ .

Finally, for each $\rho \leq r < \infty$, define

$$\Omega_r := \psi(\Delta_r), \quad L_r := \partial\Omega_r, \quad G_r := \mathbb{C} \setminus \overline{\Omega}_r,$$

so that for $r > \rho$, L_r is an analytic Jordan curve.

Carleman proved that

$$(1) \quad \lim_{n \rightarrow \infty} \frac{p_n(z)}{\sqrt{n+1}[\phi(z)]^n} = \phi'(z), \quad z \in \Omega_\rho,$$

the convergence being uniform on compact subsets of Ω_ρ (for a more complete statement, see [2, Satz IV], [5, Sec. 1], and also [3]).

This establishes the asymptotic behavior of $p_n(z)$ on the closed exterior $\overline{\Omega}_1$ of L_1 , and on a portion of its interior G_1 , namely, on the “strip” $\Omega_\rho \cap G_1$. What happens at the remaining points of G_1 has been recently investigated in [4, 10]. It turns out that there is a subset $\Sigma_1 \subset G_1$, which is, in general, larger than the strip $\Omega_\rho \cap G_1$, on which an asymptotic formula just like (1) holds true. This set Σ_1 is, however, less straightforward to define, and its construction depends on a conformal map $\varphi(z)$ of G_1 onto \mathbb{D}_1 .

Such a conformal map φ has a meromorphic and univalent continuation to $G_{1/\rho}$ (see [4] for details), so that the composition $\varphi(\psi(w))$ is a well-defined meromorphic function in the annulus $\rho < |w| < 1/\rho$. We can then define the important quantity $\mu \geq 0$ to be the smallest number such that $\varphi(\psi(w))$ has a meromorphic continuation, denoted by $h_\varphi(w)$, to the annulus $\mu < |w| < 1/\rho$.

We let Σ be the set of points $z \in G_1$ such that the equation

$$(2) \quad h_\varphi(w) = \varphi(z)$$

has at least one solution in the annulus $\mu < |w| < 1$, and let $\Sigma_0 := G_1 \setminus \Sigma$.

For fixed $z \in \Sigma$, of the solutions that the equation (2) has in $\mu < |w| < 1$, only finitely many (say s of them) have largest modulus, and we denote these solutions of largest modulus by $\omega_{z,1}, \dots, \omega_{z,s}$. Letting $\alpha_{z,k}$ denote the multiplicity of h_φ at $\omega_{z,k}$, we associate to each integer $p \geq 1$ the set

$$(3) \quad \Sigma_p := \{z \in \Sigma : \alpha_{z,1} + \dots + \alpha_{z,s} = p\}.$$

Thus, Σ_1 consists of those points $z \in \Sigma$ such that the equation (2) has one solution of largest modulus, and this solution is simple. Finally, we define the functions $\Phi : \Sigma_1 \rightarrow \{w : \mu < |w| < 1\}$ and $r : G_1 \rightarrow [\mu, 1)$ by

$$\Phi(z) := \omega_{z,1}, \quad z \in \Sigma_1,$$

and

$$r(z) := \begin{cases} |\omega_{z,1}|, & z \in \Sigma, \\ \mu, & z \in \Sigma_0. \end{cases}$$

It is not difficult to see (see Lemma 11 and Corollary 12 of [4]) that neither μ nor the Σ_p sets depend on the interior conformal map φ chosen. Moreover, Σ and Σ_1 are open, $\Phi(z)$ is analytic and univalent, and $r(z)$ is continuous.

Notice that for $z \in \Omega_\rho \cap G_1$,

$$h_\varphi(\phi(z)) = \varphi(\psi(\phi(z))) = \varphi(z),$$

so that

$$\Phi(z) = \phi(z), \quad z \in \Omega_\rho \cap G_1,$$

and therefore, $\Sigma_1 \supset \Omega_\rho \cap G_1$. In general, Σ_1 is larger than $\Omega_\rho \cap G_1$.

Using this partition of G_1 into Σ_p sets, it was proven in [4] that

$$(4) \quad \lim_{n \rightarrow \infty} \frac{p_n(z)}{\sqrt{n+1}[\Phi(z)]^n} = \Phi'(z), \quad z \in \Sigma_1,$$

the convergence being uniform on compact subsets of Σ_1 , and that

$$(5) \quad \limsup_{n \rightarrow \infty} |p_n(z)|^{1/n} = r(z), \quad z \in G_1 \setminus \Sigma_1.$$

Both (4) and (5) can be obtained from the following integral representation, which is fundamental as well for proving the results of this paper.

Proposition 1.1. *Let r be any fixed number satisfying that $\rho < r < 1$. Then, for every integer n sufficiently large,*

$$(6) \quad p_n(z) = \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_1} \frac{w^n(1+K_n(w))dw}{h_\varphi(w) - \varphi(z)}, \quad z \in G_1,$$

where $K_n(w)$ is analytic in $|w| < 1/r$ and $K_n(w) = O(r^{2n})$ locally uniformly as $n \rightarrow \infty$ on $|w| < 1/r$.

A first version of Proposition 1.1 was proven in [10]. The version stated above is simpler to use and we shall briefly indicate at the end of Section 4 below how to derive it from the recent results of [3].

Roughly speaking, (6) is telling us that for each fixed $z \in G_1$, $p_n(z)$ behaves as $n \rightarrow \infty$ like the $-(n+1)$ th coefficient of the Laurent expansion that the function $w \mapsto [h_\varphi(w) - \varphi(z)]^{-1}$ has in the annulus $\rho < |w| < 1$. If $z \in \Sigma_1$, the Laurent expansion encounters on its inner circle of convergence just one singularity, which happens to be a simple pole, thereby implying (4)¹.

If $z \in \Sigma_0$, however, the inner circle of convergence of the Laurent expansion is \mathbb{T}_μ , where the function $h_\varphi(w)$ encounters its first nonpolar singularity. Given that

¹If $z \in \Sigma_p$, $p \geq 2$, the first singularities are also finitely many poles, but they have a total multiplicity larger than 1, and although this certainly leads to a better estimate than (5), that estimate is essentially pointwise, unless more is known about the particularities of the orthogonality domain.

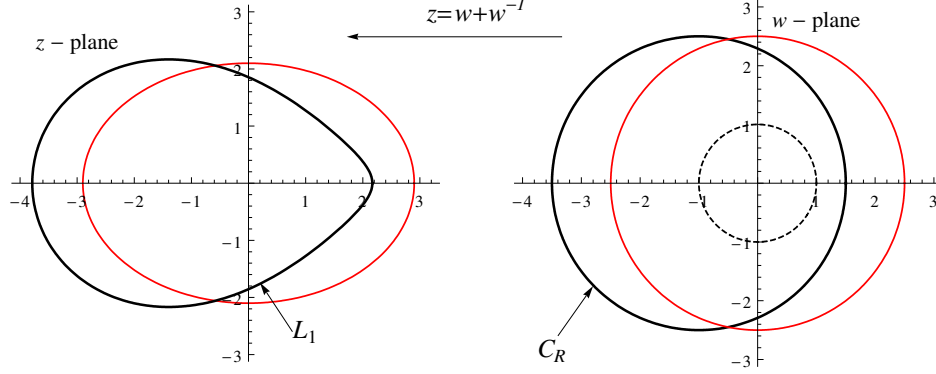


FIGURE 1. L_1 is the image by the Zhukovsky transformation of the circle $C_R = \{w : |w + 1| = 2.5\}$.

the behavior of $h_\varphi(w)$ on \mathbb{T}_μ can be wildly erratic, (5) is, in general, the best we can say for points $z \in \Sigma_0$.

Nonetheless, for more specific orthogonality domains one should be able to say more than just (5), and it is the purpose of this paper to provide a “full featured” orthogonality domain for which the corresponding set Σ_1 is *larger* than $\Omega_\rho \cap G_1$, *the interior of Σ_0 is nonempty*, and we can establish the strong asymptotic behavior of $p_n(z)$ for every point $z \in \Sigma_0$.

It turns out, however, that providing such an example is much trickier than it might seem at first sight. The difficulty lies in that, when trying to guarantee that Σ_1 be larger than $\Omega_\rho \cap G_1$, we lose control of the nature of the first *nonpolar* singularities that the function $h_\varphi(w)$ encounters.

We take for orthogonality domain G_1 the interior of the image by the Zhukovsky transformation of a circle C_R centered at -1 of radius $R > 2$ (see Figure 1). In other words, the boundary

$$(7) \quad L_1 := \{w - 1 + (w - 1)^{-1} : |w| = R\}$$

of G_1 is a level curve of the inverse of the shifted Zhukovsky transformation $w \mapsto w - 1 + (w - 1)^{-1}$.

To avoid unnecessary complications, we shall refer to Figure 2 and content ourselves with a visual understanding of the geometric aspects of the curve L_1 defined by (7), making it all precise in Proposition 2.1 of the next section.

For this curve we have that $G_1 = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where Σ_1 is the full greyish region, Σ_2 is the half-open segment $(x_\mu, 2]$, and Σ_0 is the white region together with its boundary. The set $\Omega_\rho \cap G_1$ is the strip between L_1 and the dotted line L_ρ . Also, for this curve we have

$$\mu = \frac{R - \sqrt{R^2 - 4}}{2}.$$

What makes this example intriguing is that the corresponding function $h_\varphi(w)$ encounters on the circle of radius \mathbb{T}_μ a non-isolated singularity of “essential type”,

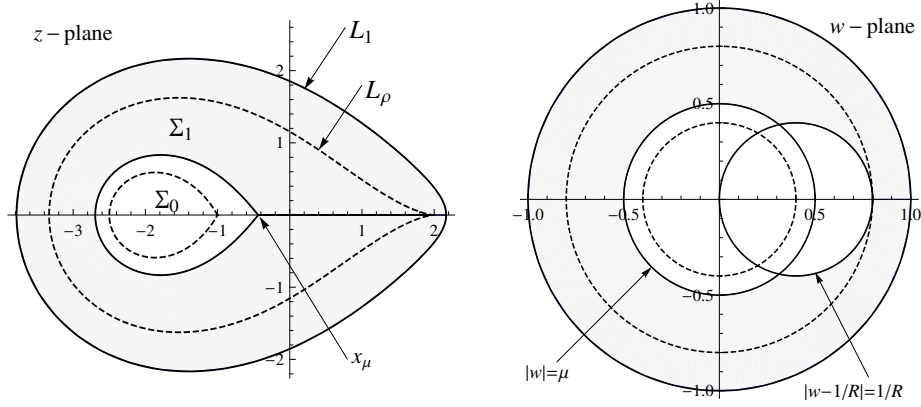


FIGURE 2. Sets Σ_1 , Σ_2 and Σ_0 for the curve L_1 defined in (7) for $R = 2.5$. Σ_1 is the greyish region, Σ_2 is the segment $(x_\mu, 2]$, and Σ_0 is the white region together with its boundary.

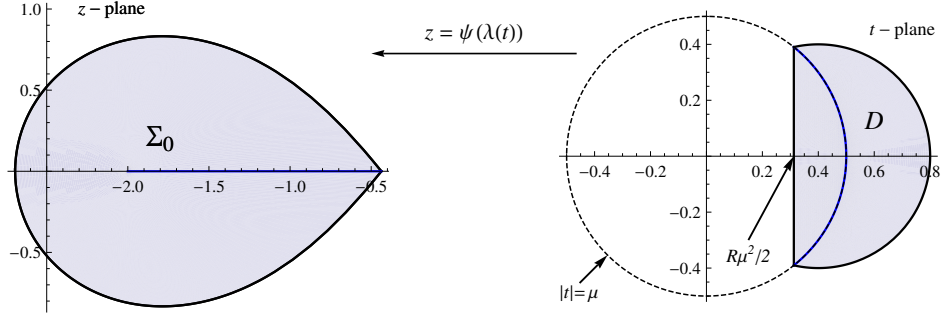


FIGURE 3. The region D is symmetric about the circle $|t| = \mu$, and is mapped by $\psi(\lambda(t))$ onto the interior of Σ_0 in a two-to-one fashion.

in the sense that in every punctured neighborhood of it, $h_\varphi(w)$ attains every value of the extended complex plane. This singularity imposes on the p_n 's a very interesting behavior, described by Theorems 1.2 and 1.3 below. We emphasize that the nature of this singularity sparks purely from geometric considerations.

Two essential components to describe the behavior of p_n on the interior of Σ_0 are the Möbius transformation $\lambda(z)$ and the doubly infinite series $\chi(t)$ defined by

$$(8) \quad \lambda(z) := \frac{z - \mu}{\mu z - 1}, \quad \chi(t) := t \sum_{k=-\infty}^{\infty} \mu^{4k} e^{(\mu - \mu^{-1})\mu^{4k}t}, \quad \Re(t) > 0.$$

For the curve L_1 given by (7), the exterior conformal map ψ of Δ_1 onto Ω_1 is given by

$$(9) \quad \psi(w) = Rw - 1 + \frac{1}{Rw - 1}, \quad w \in \overline{\mathbb{C}},$$

and the composition $\psi(\lambda(t))$ maps the region

$$(10) \quad D := \{t : |t - 1/R| < 1/R, \Re(t) > R\mu^2/2\}$$

in a two-to-one fashion onto the interior of Σ_0 (see Figure 3). The asymptotics of $p_n(z)$ for z in the interior of Σ_0 take a simpler and more elegant form if they are stated instead on the region D by means of the functions

$$\mathcal{P}_n(t) := p_n(\psi(\lambda(t))), \quad n \geq 0.$$

Hereafter the fractional part of a number x will be denoted by $\langle x \rangle$.

Theorem 1.2. *Let $\{n_k\}_{k=1}^\infty$ be a subsequence of the natural numbers. The sequence $\{\sqrt{n_k}\mu^{-n_k}\mathcal{P}_{n_k}\}_{k=1}^\infty$ converges normally on D if and only if*

$$(11) \quad \lim_{k \rightarrow \infty} e^{2\pi i(\log_{\mu^4}(n_k) - q)} = 1$$

for some $q \in [0, 1)$, in which case

$$(12) \quad \lim_{k \rightarrow \infty} \frac{\sqrt{n_k}\mathcal{P}_{n_k}(t)}{\mu^{n_k}} = f_q(t) := \frac{(1 - \mu t)^2(1 - \mu^3/t)^2}{(1 - \mu^4)R} \cdot \frac{\chi(\mu^{4q}t) - \chi(\mu^{4q+2}/t)}{t - \mu^2/t}.$$

Moreover, $f_q \neq f_p$ for $0 \leq q < p < 1$, and since the sequence $\{\langle \log_{\mu^4} n \rangle\}_{n \in \mathbb{N}}$ is dense in $[0, 1)$, it follows that the family $\{f_q(t) : 0 \leq q < 1\}$ comprises all the limit points that the sequence $\{\sqrt{n}\mu^{-n}\mathcal{P}_n\}_{n=0}^\infty$ has on D .

Thus, $p_n(z)$ decreases like μ^n/\sqrt{n} on the interior of Σ_0 , but $\lim_{n \rightarrow \infty} \sqrt{n}\mu^{-n}p_n(z)$ only exists through those subsequences $\{n_k\}$ for which $\log_{\mu^4}(n_k)$ converges modulo 1.

Theorem 1.2 follows by combining the following asymptotic formula with the integral representation for $\chi(t)$ given in (28).

Theorem 1.3. *The asymptotic representation*

$$(13) \quad \mathcal{P}_n(t) = \frac{\mu^n}{\sqrt{n}} \cdot \frac{(1 - \mu t)^2(1 - \mu^3/t)^2}{(1 - \mu^4)R} \left(\frac{\chi(nt) - \chi(n\mu^2/t)}{t - \mu^2/t} + O(1/n) \right)$$

holds true locally uniformly on D as $n \rightarrow \infty$.

Combining this theorem with (33) we obtain that

$$\frac{\sqrt{n+1}p_{n+1}(z)}{\mu^{n+1}} - \frac{\sqrt{n}p_n(z)}{\mu^n} = O(1/n)$$

locally uniformly on the interior of Σ_0 .

Remark 1. For each $z \in \mathbb{C}$, the solutions $t_{z,\pm}$ to the equation $z = \psi(\lambda(t))$ are given by

$$t_{z,\pm} = \mu \cdot \frac{4 - (R^2 - 2)z \mp R\sqrt{R^2 - 4}\sqrt{z^2 - 4}}{2(R^2 - 2 - z)},$$

which satisfy that $t_{z,+} = \mu^2/t_{z,-}$. Thus, replacing t in (12) and (13) by anyone of these two solutions yields the corresponding asymptotic statements in the z -plane, more specifically, on the interior of Σ_0 .

Remark 2. Let Z be the set of all $z_0 \in \overline{\mathbb{C}}$ with the property that every neighborhood of z_0 contains zeros of infinitely many p_n 's. From the results of [4], we infer that

$\partial\Sigma_0 \cup \Sigma_2 \subset Z \subset \Sigma_0 \cup \Sigma_2$. The points of Z lying in the interior of Σ_0 are the images under $\psi(\lambda(t))$ of the zeros that the functions

$$\frac{\chi(\mu^{4q}t) - \chi(\mu^{4q+2}/t)}{t - \mu^2/t}, \quad q \in [0, 1),$$

have in the region D . Numerical computations indicate that for at least some values of q and μ , these functions do have zeros in D .

Remark 3. The type of singularity encountered by the function h_φ of our example seems to occur with frequency. This is a consequence of the meromorphic continuation properties that h_φ displays in general, as explained in Propositions 4, 5 and 6 of [4]. Such is the case, for instance, of the level curve

$$L_1 = \left\{ z = w + \frac{1}{2w^2} : |w| = R \right\}, \quad R > 1$$

(for $R = 1$, this curve is the hypocycloid of three cusps). The example discussed in this paper, though already complex, is the simplest we have found.

Our next result establishes the behavior of $p_n(z)$ at the remaining points of G_1 .

Theorem 1.4. *The estimate*

$$(14) \quad \frac{(2R)^{n+1} p_n(z)}{\sqrt{n+1}(z+2+\sqrt{z^2-4})^n} = \frac{z+\sqrt{z^2-4}}{\sqrt{z^2-4}} + O(1/n)$$

holds uniformly as $n \rightarrow \infty$ on compact subsets of $\Sigma_1 \cup \partial\Sigma_0 \setminus \{x_\mu\}$.

For every $z = 2 \cos \theta$, $0 \leq \theta \leq \arccos(x_\mu/2)$, $x_\mu = (1 + \mu^2)^2 - 2$, we have

$$(15) \quad p_n(z) = \sqrt{n+1} (2/R)^{n+1} \cos^n(\theta/2) \left\{ \frac{\sin((n+2)\theta/2)}{2 \sin \theta} + \epsilon_n(z) \right\},$$

where $\epsilon_n(z)$ decays geometrically fast on compact subsets of $(x_\mu, 2]$, while $\epsilon_n(z) = O(1/n)$ uniformly on $[x_\mu, 2]$ as $n \rightarrow \infty$.

The rest of the paper is organized as follows. First, in Section 2 we acquire a good understanding of the meromorphic continuation properties of the associated function h_φ . This will allow us to represent the integral in (6) as an infinite, n -dependent sum of residues. In Section 3 we derive several lemmas needed to estimate the asymptotic behavior as $n \rightarrow \infty$ of such infinite sums, and finally in Section 4, we prove Theorems 1.2, 1.3 and 1.4, briefly indicating at the end how to derive Proposition 1.1 from the recent results of [3].

2. MEROMORPHIC CONTINUATION OF h_φ

For a fixed value of $R > 2$, let L_1 be given by (7). From very well-known properties of the Zhukovsky transformation $w \mapsto w + 1/w$, it follows that L_1 is an analytic Jordan curve, and that the function $\psi(w)$ given by (9) maps Δ_1 conformally onto the exterior Ω_1 of L_1 .

Moreover, ψ maps both $\{w : |w - 1/R| > 1/R\}$ and $\{w : |w - 1/R| < 1/R\}$ conformally onto $\overline{\mathbb{C}} \setminus [-2, 2]$, while mapping both the closed upper and lower halves of the circle $|w - 1/R| = 1/R$ univalently onto $[-2, 2]$. Hence,

$$\rho = 2/R,$$

and L_ρ is the image by ψ of the circle $\mathbb{T}_{2/R}$ (see Figure 2).

For every $z \in \mathbb{C}$, the equation $z = \psi(w)$ has for solutions the numbers

$$(16) \quad v_{z,+} := \frac{z + 2 + \sqrt{z^2 - 4}}{2R}, \quad v_{z,-} := \frac{z + 2 - \sqrt{z^2 - 4}}{2R}, \quad z \in \mathbb{C},$$

where we denote by $\sqrt{z^2 - 4}$ the branch of the square root of $z^2 - 4$ in $\mathbb{C} \setminus [-2, 2]$ that is positive along $(2, \infty)$, extended to $[-2, 2]$ by taking its boundary values from the upper half plane.

When $z \in \mathbb{C} \setminus [-2, 2]$, $v_{z,+}$ and $v_{z,-}$ lie, respectively, outside and inside the circle $|w - 1/R| = 1/R$, and consequently

$$|v_{z,+}| = \left| R^{-1} + \frac{R^{-2}}{v_{z,-} - R^{-1}} \right| = \left| \frac{R^{-1}}{v_{z,-} - R^{-1}} \right| |v_{z,-}| > |v_{z,-}|,$$

that is, $|v_{z,+}| > |v_{z,-}|$ for every $z \in \mathbb{C} \setminus [-2, 2]$. Of course, if $z \in [-2, 2]$, then $v_{z,+} = \bar{v}_{z,-}$, they lie on the circle $|w - 1/R| = 1/R$, and $v_{z,+} = v_{z,-}$ if and only if $z = \pm 2$.

It follows that the inverse of $\psi(w)$ is the function

$$\phi(z) = v_{z,+}, \quad z \in \Omega_\rho,$$

which is indeed analytic and univalent all over $\mathbb{C} \setminus [-2, 2]$.

As for the corresponding Σ_p sets and number μ , the following result was already obtained in [4, Theorem 10]. We shall, however, give a new proof of it that is based on finding all the solutions of the equation (2).

Proposition 2.1. *For the domain G_1 bounded by the curve L_1 of (7),*

$$(17) \quad \mu = \frac{R - \sqrt{R^2 - 4}}{2}$$

and Σ consists of those points $z \in G_1$ for which $|v_{z,+}| > \mu$. Furthermore, if $z \in \Sigma$ and ω is one of the solutions of largest modulus that the equation (2) has in $\mu < |\omega| < 1$, then $\omega \in \{v_{z,+}, v_{z,-}\}$. As a consequence,

$$G_1 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0,$$

with Σ_1 being the image by ψ of those points of \mathbb{D}_1 that lie exterior to both the circle $|w| = \mu$ and the circle $|w - 1/R| = 1/R$, and $\Sigma_2 = (x_\mu, 2]$, with $x_\mu = (1 + \mu^2)^2 - 2$ (see Figure 2).

To prove Proposition 2.1 we shall use the following fundamental result. Hereafter μ is given by (17) and $\lambda(z)$ is defined by (8). We shall implicitly use that $\lambda(z)$ is its own inverse and that

$$(18) \quad \mu = (R - \mu)^{-1}.$$

Proposition 2.2. *Let φ be a conformal map of G_1 onto \mathbb{D}_1 .*

- (a) *The function $\varphi(\psi(w))$, originally defined on $\rho < |w| < 1/\rho$, admits a meromorphic continuation, denoted by $h_\varphi(w)$, to all of $\mathbb{C} \setminus \{\mu, 1/\mu\}$. Moreover, μ and $1/\mu$ are both non-isolated singularities of h_φ of “essential type”, in the sense that in every punctured neighborhood of either one of these two points, the function h_φ attains every value of the extended complex plane.*

(b) $h_\varphi \circ \lambda$ is meromorphic in $\mathbb{C} \setminus \{0\}$, and for all $k \in \mathbb{Z} \setminus \{0\}$,

$$(19) \quad (h_\varphi \circ \lambda)(t) = \begin{cases} \frac{1}{\overline{(h_\varphi \circ \lambda)(\bar{t}/\mu^{2k})}}, & \mu^{2k+2} \leq |t| \leq \mu^{2k}, \quad k \text{ odd}, \\ (h_\varphi \circ \lambda)(t/\mu^{2k}), & \mu^{2k+2} \leq |t| \leq \mu^{2k}, \quad k \text{ even}. \end{cases}$$

(c) For every t_0 with $\mu^2 < |t_0| < 1$, the solutions that the equation

$$(h_\varphi \circ \lambda)(t) = (h_\varphi \circ \lambda)(t_0)$$

has in $0 < |t| < 1$ are the elements of the two sequences $\{\mu^{4k}t_0\}_{k=0}^\infty$ and $\{\mu^{4k+2}/t_0\}_{k=0}^\infty$. Moreover,

$$(20) \quad (h_\varphi \circ \lambda)'(\mu^{4k}t_0) = -\frac{\varphi'(\psi(\lambda(t_0)))(1-\mu^4)^2(t_0 - \mu^2/t_0)}{\mu^{4k+1}t_0(1-\mu t_0)^2(1-\mu^3/t_0)^2}, \quad k \geq 0,$$

and if $\mu < |\lambda(t_0)| < 1$, then

$$(21) \quad |\lambda(t_0)| > |\lambda(\mu^{4k}t_0)|, \quad k \geq 1.$$

Proof. Observe first that $v_{z,+}$ and $\bar{v}_{z,-}$ are reflections of each other about the circle $|w - 1/R| = 1/R$, given that $Rv_{z,+} - 1$ and $R\bar{v}_{z,-} - 1$ are reflections of each other about the unit circle. Since the reflection of the unit circle about $|w - 1/R| = 1/R$ is the circle $|w - R/(R^2 - 1)| = 1/(R^2 - 1)$, we then have that ψ maps

$$\mathfrak{D} := \{w : |w - R/(R^2 - 1)| > 1/(R^2 - 1), |w| < 1\}$$

in a two-to-one fashion onto G_1 , and $\partial\mathfrak{D}$ onto L_1 . Because φ is analytic in a neighborhood of \bar{G}_1 , the composition $h_\varphi(w) = \varphi(\psi(w))$ makes sense and is analytic in $\bar{\mathfrak{D}}$.

On the other hand, using (18), it is easy to see that λ maps the annulus $\mu^2 \leq |t| \leq 1$ conformally onto $\bar{\mathfrak{D}}$. In effect, λ is an automorphism of the unit circle, it maps $|t| = \mu$ onto $|w - 1/R| = 1/R$, and preserves reflections about circles, so that it maps $|t| = \mu^2$ onto $|w - R/(R^2 - 1)| = 1/(R^2 - 1)$. Thus, $h_\varphi \circ \lambda$ is analytic on $\mu^2 \leq |t| \leq 1$, mapping the boundary of this annulus onto the unit circle, and since

$$(\psi \circ \lambda)(\mu^2 t) = \overline{(\psi \circ \lambda)(t)} = (\psi \circ \lambda)(\bar{t}), \quad |t| = 1,$$

we then have

$$(h_\varphi \circ \lambda)(\mu^2 t) = \frac{1}{\overline{(h_\varphi \circ \lambda)(\bar{t})}}, \quad |t| = 1.$$

Hence, we can extend $h_\varphi \circ \lambda$ meromorphically to all of $\mathbb{C} \setminus \{0\}$ as specified by (19). Now, we see from (19) that $h_\varphi \circ \lambda$ maps $\mu^2 \leq |t| \leq 1$ onto $\bar{\mathbb{D}}_1$ and $\mu^4 \leq |t| \leq \mu^2$ onto $\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_1$, and that

$$(22) \quad (h_\varphi \circ \lambda)(\mu^4 t) = (h_\varphi \circ \lambda)(t).$$

Hence, $h_\varphi \circ \lambda$ maps every annulus $\mu^{4(k+1)} \leq |t| \leq \mu^{4k}$, $k \in \mathbb{Z}$, onto $\bar{\mathbb{C}}$, and thus composing back with λ (which is its own inverse) we obtain Part (a) of the proposition.

The first statement of Part (c) equally follows from (19), while from (22) we get

$$(h_\varphi \circ \lambda)'(\mu^{4k}t_0) = \mu^{-4k}(h_\varphi \circ \lambda)'(t_0) = \mu^{-4k}\varphi'(\psi(\lambda(t_0)))(\psi \circ \lambda)'(t_0), \quad j = 1, 2,$$

which combined with

$$(\psi \circ \lambda)'(t) = \left[\mu^{-1} \frac{t - \mu^3}{\mu t - 1} + \mu \frac{\mu t - 1}{t - \mu^3} \right]' = - \frac{(1 - \mu^4)^2 (t - \mu^2/t)}{\mu t (1 - \mu t)^2 (1 - \mu^3/t)^2}$$

yields (20).

Finally, (21) follows from the fact that $\lambda(t)$ maps $[-\infty, 0]$ onto $[\mu, 1/\mu]$ and $[0, \infty]$ onto $[-\infty, \mu] \cup [1/\mu, \infty]$, respectively, while any other line passing through the origin gets mapped onto a circle passing through the points μ and $1/\mu$, and the two points of this circle that are closest to and farthest from the origin lie, respectively, inside \mathbb{T}_μ and outside \mathbb{T}_1 . \square

We can now give the

Proof of Proposition 2.1. The fact that μ as given by (17) is the smallest number such that $\varphi(\psi(w))$ admits a meromorphic continuation to the annulus $\mu < |w| < 1$ now emerges clearly from Proposition 2.2(a).

For a given $z \in G_1$, we now look for the solutions that the equation (2) has on $\{w : |w| < 1, w \neq \mu\}$. Since $\lambda(t)$ is an automorphism of the unit disk, this is equivalent to finding the solutions that the equation

$$(23) \quad (h_\varphi \circ \lambda)(t) = \varphi(z)$$

has on $0 < |t| < 1$. We have already observed in the proof of Proposition 2.2 that $\psi(w)$ maps the region \mathfrak{D} onto G_1 , and that for each $z \in G_1$, the solutions to $z = \psi(\lambda(t))$ are the numbers $v_{z,\pm}$. Since φ is univalent on G_1 and $\lambda(t)$ maps $\mu^2 < |t| < 1$ onto \mathfrak{D} , we see that (23) has exactly two solutions on $\mu^2 < |t| < 1$, given by $\lambda(v_{z,\pm})$ (recall λ is its own inverse). Since $\lambda(v_{z,+}) = \mu^2/\lambda(v_{z,-})$, we infer from Proposition 2.2(c) that the solutions that the equation $h_\varphi(w) = \varphi(z)$ has in $\mathbb{D}_1 \setminus \{\mu\}$ are the elements of the two sequences $\{\lambda(\mu^{4k}\lambda(v_{z,\pm}))\}_{k=0}^\infty$.

Now, $\lambda(t)$ maps the disk $|t - 1/R| \leq 1/R$ onto $|w| \leq \mu$, and since $|v_{z,+}| \geq |v_{z,-}|$, we find that $h_\varphi(w) = \varphi(z)$ has solutions in $\mu < |w| < 1$ if and only if $|v_{z,+}| > \mu$, in which case, by (21), the solutions of largest modulus are at most two and contained in $\{v_{z,+}, v_{z,-}\}$. \square

3. AUXILIARY LEMMAS

The following lemmas have been set apart because they are rather technical and may obscure the central idea of the proof of Theorems 1.3 and 1.4. For a first reading, we recommend the reader to trust the validity of Lemma 3.3 below and move on to the next section.

Lemma 3.1. *For every compact set $E \subset \{t : |1 - Rt| < 1\}$, there exist positive constants m and M such that for every integer $n \geq 1$,*

$$\left| e^{-(\mu^{-1}-\mu)ts} - \mu^{-n} \lambda^n(ts/n) \right| \leq \frac{Ms^2 e^{-ms}}{n}, \quad t \in E, \quad 0 \leq s \leq n.$$

Proof. For every $z \in \mathbb{C}$, the function $\kappa(s) := |1 - sz|$ is convex in \mathbb{R} . Hence, for $z \in U_1 := \{z : |1 - z| < 1\}$ and an integer $n \geq 1$, we have

$$(24) \quad \left| 1 - \frac{sz}{n} \right| \leq \left| 1 - (1 - |1 - z|) \frac{s}{n} \right| \leq e^{-(1-|1-z|)s/n}, \quad 0 \leq s \leq n.$$

Next, suppose t is such that $|\mu^{-1}\lambda(t)| < 1$ and consider the Möbius transformation

$$\sigma_t(s) := \frac{(1 - \mu^2)t}{\mu(1 - \mu ts)}, \quad s \in \mathbb{R}.$$

Since

$$(25) \quad \mu^{-1}\lambda(t) = 1 - \frac{(1 - \mu^2)t}{\mu(1 - \mu t)},$$

we readily see that $\sigma_t(1) \in U_1$. Also, since $|\mu^{-1}\lambda(t)| < 1$ if and only if $|1 - Rt| < 1$, and $\mu^{-1}(1 - \mu^2) = (1 - \mu^2)(1 + \mu^2)^{-1}R < R$, it follows that $\sigma_t(0) = \mu^{-1}(1 - \mu^2)t \in U_1$. Then, given that $\sigma_t(\infty) = 0 \in \partial U_1$ and that σ maps the real line conformally onto a circle, we conclude that σ_t must map the segment $[0, 1]$ onto a circular arc that lies inside U_1 . Therefore, by (24) and (25), we have for all $0 \leq s \leq n$ and $|1 - Rt| < 1$ that

$$(26) \quad |\mu^{-1}\lambda(ts/n)| = \left| 1 - \frac{s}{n} \sigma_t(s/n) \right| \leq e^{-(1 - |1 - \sigma_t(s/n)|)s/n}.$$

Now, to abbreviate, set $\alpha = \mu^{-1} - \mu$, so that for $0 \leq s \leq n$ and $|1 - Rt| < 1$ we have

$$(27) \quad \begin{aligned} \left| e^{-\alpha ts/n} - \mu^{-1}\lambda(ts/n) \right| &= \left| e^{-\alpha ts/n} - \left(1 - \frac{\alpha ts/n}{1 - \mu ts/n} \right) \right| \\ &= \frac{s^2}{n^2} \left| \frac{\mu \alpha t^2}{1 - \mu ts/n} + \sum_{j=2}^{\infty} \frac{(-\alpha t)^j}{j!} \left(\frac{s}{n} \right)^{j-2} \right| \\ &\leq \frac{s^2}{n^2} \left(\frac{\mu \alpha |t|^2}{1 - \mu |t|} + e^{\alpha |t|} - \alpha |t| - 1 \right). \end{aligned}$$

Thus, for a compact set $E \subset \{t : |1 - Rt| < 1\}$, and with the constants

$$\begin{aligned} M &:= \max_{t \in E} \left(\frac{\mu \alpha |t|^2}{1 - \mu |t|} + e^{\alpha |t|} - \alpha |t| - 1 \right), \quad m_1 := \alpha \min_{t \in E} \Re(t) > 0, \\ m_2 &:= \min_{(t,s) \in E \times [0,n]} (1 - |1 - \sigma_t(s/n)|) > 0, \quad m := 2^{-1} \min\{m_1, m_2\}, \end{aligned}$$

we obtain from (26) and (27) that

$$\begin{aligned} |e^{-\alpha ts} - \mu^{-n}\lambda^n(ts/n)| &\leq \frac{Ms^2}{n^2} \sum_{\ell=1}^n e^{-\alpha \Re(t)s(\ell-1)/n} |\mu^{-1}\lambda(ts/n)|^{n-\ell} \\ &\leq \frac{Ms^2}{n^2} \sum_{\ell=1}^n e^{-\alpha \Re(t)s(\ell-1)/n} e^{-(1 - |1 - \sigma_t(s/n)|)s(n-\ell)/n} \\ &\leq \frac{Ms^2 e^{-ms}}{n}, \quad t \in E, \quad 0 \leq s \leq n. \end{aligned}$$

□

Recall that $\langle x \rangle$ denotes the fractional part of x . It is easy to verify that

$$(28) \quad \chi(\gamma t) = \frac{\mu^3 t^2}{1 + \mu^2} \int_0^\infty \mu^{-4(\log_{\mu^4}(s/\gamma))} s e^{(\mu - \mu^{-1})ts} ds, \quad \Re(t) > 0, \quad \gamma > 0.$$

In particular, we see that the functions $\{\chi(nt)\}_{n=1}^\infty$ are uniformly bounded on compacts of $\Re(t) > 0$.

Lemma 3.2. *With $G_n(t) := \mu^{-n}\lambda^n(t)\lambda'(t)$, $n \geq 1$, and $\chi(t)$ defined by (8), we have*

$$(29) \quad \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s G'_{n+1}(st/n) ds = \frac{(n+1)(1-\mu^2)(1-\mu^4)[\chi(nt) + O(1/n)]}{\mu^4 t^2}$$

locally uniformly on $\{t : |1 - Rt| < 1\}$ as $n \rightarrow \infty$.

Proof. Let $E \subset \{t : |1 - Rt| < 1\}$ be compact. First we notice that

$$(30) \quad G'_{n+1}(st/n) = (n+1)\mu^{-n-1}\lambda^n(ts/n)\mathfrak{L}(st/n)$$

with

$$(31) \quad \begin{aligned} \mathfrak{L}(st/n) &= \frac{(1-\mu^2)^2}{(1-\mu ts/n)^4} + \frac{2\mu(1-\mu^2)(ts/n-\mu)}{(n+1)(1-\mu ts/n)^4} \\ &= (1-\mu^2)^2 [1 + O(s/n) + O(1/n)] \end{aligned}$$

uniformly for $(t, s) \in E \times [0, n]$ as $n \rightarrow \infty$. Hence, with $\alpha = \mu^{-1} - \mu$ and given that $1 \leq \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} \leq \mu^{-4}$, we get

$$(32) \quad \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s e^{-\alpha ts} \mathfrak{L}(st/n) ds = (1-\mu^2)^2 \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s e^{-\alpha ts} ds + O(1/n)$$

uniformly for $(t, s) \in E \times [0, n]$ as $n \rightarrow \infty$.

Combining (30), (31), (32) and Lemma 3.1, we readily see that there exist positive constants M' and m such that

$$\begin{aligned} & \left| \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s G'_{n+1}(st/n) ds - \frac{(n+1)(1-\mu^2)^2}{\mu} \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s e^{-\alpha ts} ds \right| \\ & \leq \frac{n+1}{\mu} \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s |\mu^{-n}\lambda^n(ts/n) - e^{-\alpha ts}| |\mathfrak{L}(st/n)| ds + O(1) \\ & \leq M' \int_0^\infty s^3 e^{-ms} ds + O(1) \end{aligned}$$

uniformly in $t \in E$ as $n \rightarrow \infty$, which together with (28) (take $\gamma = n$) yields (29). \square

Lemma 3.3. *For $G_n(t)$ defined as in Lemma 3.2, we have*

$$\sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k}t) = - \frac{(n+1)(1-\mu^2) [\chi(nt) + O(1/n)]}{n^2 t}$$

locally uniformly on $\{t : |1 - Rt| < 1\}$ as $n \rightarrow \infty$.

Proof. Using summation by parts we find

$$\begin{aligned} \sum_{k=0}^K \mu^{4k} G_{n+1}(\mu^{4k}t) &= \frac{G_{n+1}(t)}{1-\mu^4} - \frac{\mu^{4(K+1)} G_{n+1}(\mu^{4K}t)}{1-\mu^4} \\ &\quad + \sum_{k=0}^{K-1} \frac{\mu^{4(k+1)}}{1-\mu^4} [G_{n+1}(\mu^{4(k+1)}t) - G_{n+1}(\mu^{4k}t)]. \end{aligned}$$

Letting $K \rightarrow \infty$ and using Lemma 3.2 we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k}t) &= \frac{G_{n+1}(t)}{1-\mu^4} + \sum_{k=0}^{\infty} \frac{\mu^{4(k+1)}}{1-\mu^4} \int_{\mu^{4k}}^{\mu^{4(k+1)}} \frac{\partial G_{n+1}(st)}{\partial s} ds \\ &= \frac{G_{n+1}(t)}{1-\mu^4} - \frac{\mu^4 t}{(1-\mu^4)n^2} \int_0^n \mu^{-4(\log_{\mu^4}(s/n))} s G'_{n+1}(st/n) ds \\ &= - \frac{(n+1)(1-\mu^2) [\chi(nt) + O(1/n)]}{n^2 t} \end{aligned}$$

locally uniformly on $\{t : |1 - Rt| < 1\}$ as $n \rightarrow \infty$. \square

Before passing to the next section, we observe that

$$(33) \quad \chi(nt) - \chi((n+1)t) = O(1/n)$$

locally uniformly on $\Re(t) > 0$. Indeed, from the representation (28) we find

$$\begin{aligned} \chi(nt) - \chi((n+1)t) &= \frac{\mu^3 t^2}{1 + \mu^2} \int_0^\infty \mu^{-4(\log_{\mu^4}(s/n))} s e^{-\alpha t s} \left(1 - \frac{(n+1)^2}{n^2} e^{-\alpha t s/n} \right) ds, \end{aligned}$$

and the claim follows since

$$|1 - e^{-\alpha t s/n}| = \frac{\alpha s}{n} \left| \int_0^t e^{-\alpha s z/n} dz \right| \leq \frac{\alpha s |t|}{n}, \quad \Re(t) > 0,$$

where the integration is taken along the segment from 0 to t .

4. PROOFS

Proof of Theorems 1.3 and 1.4. We have already observed in the proof of Proposition 2.2 that the function $\psi(\lambda(t))$ maps the annulus $\mu^2 < |t| < 1$ onto G_1 , and that for every $z \in G_1$, the only solutions that the equation $z = \psi(\lambda(t))$ has in said annulus are the numbers $t_{z,\pm} = \lambda(v_{z,\pm})$, which satisfy $t_{z,+} = \mu^2/t_{z,-}$. If we now make the change of variables $w = \lambda(\zeta)$ in the integral representation given by Proposition 1.1, use Proposition 2.2(c) and the residue theorem, we get that for every integer $N \geq 1$, $z \in G_1 \setminus \{-2, 2\}$, and n sufficiently large,

$$\begin{aligned} &(n+1)^{-1/2} p_n(z) \\ &= \frac{\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_1} \frac{[\lambda(\zeta)]^n \lambda'(\zeta) [1 + K_n(\lambda(\zeta))] d\zeta}{(h_\varphi \circ \lambda)(\zeta) - (h_\varphi \circ \lambda)(t_{z,\pm})} \\ &= \frac{(1 - \mu t_{z,\pm})^2 (1 - \mu t_{z,\mp})^2 \mu^{n+1}}{(1 - \mu^4)^2 (t_{z,\pm} - t_{z,\mp})} \left[\zeta \sum_{k=0}^{N-1} \mu^{4k} G_n(\mu^{4k} \zeta) [1 + K_n(\lambda(\mu^{4k} \zeta))] \right]_{\zeta=t_{z,\pm}}^{\zeta=t_{z,\mp}} \\ &\quad + \frac{\varphi'(z)}{2\pi i} \int_{|\zeta|=\mu^{4N-2}} \frac{[\lambda(\zeta)]^n \lambda'(\zeta) [1 + K_n(\lambda(\zeta))] d\zeta}{(h_\varphi \circ \lambda)(\zeta) - (h_\varphi \circ \lambda)(t_{z,\pm})}, \end{aligned}$$

where $G_n(\zeta) = \mu^{-n} \lambda^n(\zeta) \lambda'(\zeta)$, and $K_n(\lambda(t)) = O(r^{2n})$ uniformly on $|t| \leq 1$ as $n \rightarrow \infty$ for every $r \in (\rho, 1)$. Here we are using the notation $[F(\zeta)]_{\zeta=a}^{\zeta=b} = F(b) - F(a)$.

Since $|\lambda(\zeta)| = 1$ for $|\zeta| = 1$ and $|(h_\varphi \circ \lambda)(\zeta)| = 1$ for $|\zeta| = \mu^{4N-2}$, we have

$$\left| \frac{1}{2\pi i} \int_{|\zeta|=\mu^{4N-2}} \frac{[\lambda(\zeta)]^n \lambda'(\zeta) [1 + K_n(\lambda(\zeta))] d\zeta}{(h_\varphi \circ \lambda)(\zeta) - (h_\varphi \circ \lambda)(t_{z,\pm})} \right| \leq \frac{O(\mu^{4N})}{1 - |\varphi(z)|} \xrightarrow{N \rightarrow \infty} 0,$$

and so we arrive at the following representation, valid for all $z \in G_1 \setminus \{-2, 2\}$ and n large:

$$(34) \quad (n+1)^{-1/2} p_n(z) = \frac{(1 - \mu t_{z,\pm})^2 (1 - \mu t_{z,\mp})^2 \mu^{n+1}}{(1 - \mu^4)^2 (t_{z,\pm} - t_{z,\mp})} \left[\zeta \sum_{k=0}^{\infty} \mu^{4k} G_n(\mu^{4k} \zeta) [1 + O(r^{2n})] \right]_{\zeta=t_{z,\pm}}^{\zeta=t_{z,\mp}},$$

where the constant involved in the $O(r^{2n})$ term above is independent of n and z .

In the above calculations, the restriction that $z \notin \{-2, 2\}$ is a technical one to avoid dealing with double poles in the residue computations. But of course, by the analyticity of the functions involved, the same estimates remain true when letting $z \rightarrow \pm 2$.

Notice that since $\lambda(t)$ maps the disk $\{t : |t - 1/R| < 1/R\}$ onto $|w| < \mu$, we have that for $k \geq 0$ and t lying in that disk,

$$(35) \quad |G_n(\mu^{4k} t)| = \mu^{-n} |\lambda(\mu^{4k} t)|^n |\lambda'(\mu^{4k} t)| \leq \max_{|t-1/R| \leq 1/R} |\lambda'(t)|.$$

Now, it is easy to verify that the function $\psi(\lambda(t))$ maps the region D defined by (10) onto the interior of Σ_0 , that $D \subset \{t : \mu^2 < |t| < 1\}$, and that D is symmetric with respect to the circle $|t| = \mu$. Hence, if we make the replacement $t = t_{z,\pm}$ in (34), then $t_{z,\mp} = \mu^2/t$ and $z = \psi(\lambda(t))$, and Theorem 1.3 follows immediately by combining (34), (35), (33), and Lemma 3.3.

We now prove the equality (15) in Theorem 1.4, which we do locally by showing that every $z_0 \in [x_\mu, 2]$ has a neighborhood $B_\epsilon(z_0) := \{z : |z - z_0| < \epsilon\}$ such that (15) holds uniformly on $z \in B_\epsilon(z_0) \cap [x_\mu, 2]$. The proof of (14) is omitted as it follows in a very similar way.

First, observe that as z varies over the interval $[x_\mu, 2]$, the two solutions $v_{z,\pm}$ of the equation $z = \psi(w)$ satisfy $v_{z,+} = \bar{v}_{z,-}$ and they vary along that arc of the circle $|w - 1/R| = 1/R$ that lies on $\mu \leq |w| < 1$. Parametrizing the upper half of this arc by setting

$$(36) \quad v_{z,+} = R^{-1} + R^{-1} e^{i\theta} = 2R^{-1} e^{i\theta/2} \cos(\theta/2), \quad 0 \leq \theta \leq \arccos(x_\mu/2),$$

we see that $|v_{z,+}| = 2R^{-1} |\cos(\theta/2)|$ varies from μ to $2/R$, and equals μ exactly when $z = x_\mu$.

For a point $z_0 \in (x_\mu, 2] = \Sigma_2$, we have that $|v_{z_0,\pm}| = |\lambda(t_{z_0,\pm})| > \mu$. In view of the inequality (21) of Proposition 2.2(c), this allows us to find numbers ϱ_1 and ϱ_2 such that

$$|\lambda(\mu^{4k} t_{z_0,\pm})| < \varrho_1 < \varrho_2 < |v_{z_0,\pm}|, \quad k \geq 1.$$

Hence, if $\epsilon > 0$ is chosen sufficiently small, we can guarantee that

$$(37) \quad |\lambda(\mu^{4k} t_{z,\pm})| < \varrho_1 < \varrho_2 < |v_{z,\pm}|, \quad z \in B_\epsilon(z_0), \quad k \geq 1,$$

so that the dominant term in the right-hand side of (34) corresponds to $k = 0$ and we get from that for all $z \in B_\epsilon(z_0)$

$$(38) \quad \begin{aligned} \frac{p_n(z)}{\sqrt{n+1}} &= \frac{(1 - \mu t_{z,\pm})^2 (1 - \mu t_{z,\mp})^2 \mu (t_{z,\mp} \lambda'(t_{z,\mp}) v_{z,\mp}^n - t_{z,\pm} \lambda'(t_{z,\pm}) v_{z,\pm}^n)}{(1 - \mu^4)^2 (t_{z,\pm} - t_{z,\mp})} \\ &\quad + O(\varrho_2^n r^{2n}) + O(\varrho_1^n) \\ &= \frac{v_{z,+}^n}{\psi'(v_{z,+})} + \frac{v_{z,-}^n}{\psi'(v_{z,-})} + O(\varrho_2^n r^{2n}) + O(\varrho_1^n) \end{aligned}$$

uniformly for $z \in B_\epsilon(z_0)$ as $n \rightarrow \infty$. To get this last equality we have used (20), and in case $z = z_0 = 2$, the estimate is to be understood in a limiting sense.

If $z_0 = x_\mu$, we have $|v_{z_0, \pm}| = \mu$ and the points $t_{z_0, \pm}$ are located precisely where the circles $|t| = \mu$ and $|t - 1/R| = 1/R$ intersect at. Hence, if we pick $\epsilon > 0$ sufficiently small, we can guarantee that as z varies over $B_\epsilon(x_\mu)$, the points $\mu^4 t_{z, \pm}$ vary over some fixed compact subset of $|t - 1/R| < 1/R$, so that (35) holds true for $t = t_{z, \pm}$ and $k \geq 1$, and we get once again from (34) and Lemma 3.3 that

$$(39) \quad (n+1)^{-1/2} p_n(z) = \frac{v_{z,+}^n}{\psi'(v_{z,+})} + \frac{v_{z,-}^n}{\psi'(v_{z,-})} + O(\mu^n/n)$$

uniformly for $z \in B_\epsilon(x_\mu)$ as $n \rightarrow \infty$. Using (36) and (37) one can easily verify that for $z \in B_\epsilon(x_\mu) \cap [x_\mu, 2]$, (38) and (39) transform into (15). \square

Proof of Theorem 1.2. In view of Theorem 1.3, it suffices to show that for a subsequence $\{n_k\} \subset \mathbb{N}$, the functions $\chi(n_k t)$ converge normally on D as $k \rightarrow \infty$ if and only if (11) holds true for some $q \in [0, 1)$.

The “if” part, that is, $\chi(n_k t) \rightarrow \chi(\mu^{4q} t)$ whenever (11) holds, follows directly from the representation (28). For the “only if” part, suppose that the functions $\chi(n_k t)$ converge as $k \rightarrow \infty$, but that (11) holds true for no $q \in [0, 1)$. Then, we can find two subsequences of $\{n_k\}$, say \mathcal{N}_1 and \mathcal{N}_2 , such that

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}_1}} e^{2\pi i \log_{\mu^4}(n)} = e^{2\pi i q}, \quad \lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}_2}} e^{2\pi i \log_{\mu^4}(n)} = e^{2\pi i p}$$

for some $0 \leq q < p < 1$. But this and the convergence of $\chi(n_k t)$ leads to a contradiction, for we shall now prove that

$$\chi(\mu^{4q} t) - \chi(\mu^{4q+2}/t) \neq \chi(\mu^{4p} t) - \chi(\mu^{4p+2}/t)$$

(or equivalently, that $f_q \neq f_p$) for $0 \leq q < p < 1$.

Making $t = \mu^{4x+1}$, $\varrho = \mu^4$, $\mathfrak{q} = q + 1/4$, and defining

$$g_{\mathfrak{q}}(x) := \chi(\varrho^{\mathfrak{q}+x}) - \chi(\varrho^{\mathfrak{q}-x}), \quad x \in \mathbb{R},$$

we see that it suffices to show that $g_{\mathfrak{q}} - g_{\mathfrak{p}} \neq 0$ for $1/4 \leq \mathfrak{q} < \mathfrak{p} < 5/4$. The function

$$g_{\mathfrak{q}}(x) = \sum_{k=-\infty}^{\infty} \varrho^{k+\mathfrak{q}+x} e^{-\alpha \varrho^{k+\mathfrak{q}+x}} - \sum_{k=-\infty}^{\infty} \varrho^{k+\mathfrak{q}-x} e^{-\alpha \varrho^{k+\mathfrak{q}-x}}, \quad \alpha = \mu^{-1} - \mu > 0,$$

is analytic and 1-periodic on \mathbb{R} . Using the Poisson summation formula, we find its Fourier expansion to be

$$g_{\mathfrak{q}}(x) - g_{\mathfrak{p}}(x) = 8 \sum_{n=1}^{\infty} \Re(c_n) \sin(\pi n(\mathfrak{p} - \mathfrak{q})) \sin(2\pi n x), \quad x \in \mathbb{R},$$

where

$$c_n = \int_{-\infty}^{\infty} \varrho^t e^{-\alpha \varrho^t} e^{i\pi n(2t - \mathfrak{q} - \mathfrak{p})} dt,$$

and we only need to show that there is at least one integer $n \geq 1$ with $\Re(c_n) \neq 0$.

Making the change of variable $v = \alpha \varrho^t$ and using Stirling's approximation formula for the Gamma function we find

$$\begin{aligned} c_n &= \frac{e^{-i\pi n(\mathfrak{q}+\mathfrak{p})} \Gamma(1 + 2\pi n i / \ln \varrho)}{\alpha^{1+2\pi n i / \ln \varrho} \ln \varrho} \\ &= \frac{|\Gamma(1 + 2\pi n i / \ln \varrho)| (1 + O(1/n))}{\alpha \ln \varrho} \cdot e \left(\frac{n \ln n}{\ln \varrho} - \sigma n + \frac{1}{8} \right), \quad n \geq 1, \end{aligned}$$

where $e(x) := \exp(2\pi i x)$ and

$$\sigma := \frac{\mathfrak{q} + \mathfrak{p}}{2} + \frac{1 + \ln \alpha - \ln(2\pi / \ln \varrho)}{\ln \varrho}.$$

It follows that $c_n \neq 0$ for all $n \geq 1$. Moreover, taking the quotient c_{n+1}/c_n , we find that if all the c_n 's are purely imaginary, then

$$\Im \left\{ e \left(\frac{\ln(1 + \frac{1}{n})^n}{\ln \varrho} + \frac{\ln(n+1)}{\ln \varrho} - \sigma \right) (1 + O(1/n)) \right\} = 0, \quad n \geq 1,$$

contradicting the fact that the sequence $\{\ln n / \ln \varrho\}_{n=1}^\infty$ is dense in $[0, 1]$ modulo 1.

This last statement follows from the more general and easily verifiable observation that $\{\langle a_n \rangle\}_{n=1}^\infty$ is dense in $[0, 1]$ whenever $\lim_{n \rightarrow \infty} a_n = +\infty$ and $\lim_{n \rightarrow \infty} a_{n+1} - a_n = 0$. \square

We finish this section by briefly indicating how to obtain Proposition 1.1 from the recent results of [3].

Let κ_n denote the leading coefficient of p_n , and let $P_n(z) = \kappa_n^{-1} p_n(z)$ be the n th monic orthogonal polynomial. Carleman proved in [2, Satz IV] that

$$(40) \quad \kappa_n = \sqrt{n+1} [\phi'(\infty)]^{n+1} (1 + O(\rho^{2n}))$$

as $n \rightarrow \infty$.

Let r be a number such that $\rho < r < 1$. For this r and every integer $n \geq 0$, we construct a sequence of functions $\{f_{n,k}(z)\}_{k=0}^\infty$ as specified by equations (8) and (9) of [3]. In [3, Proof of Theorem 1.1], it is shown that for n large enough, the series $\sum_{k=0}^\infty f_{n,2k+2}(z)$ and $\sum_{k=0}^\infty f_{n,2k+1}(z)$ converge absolutely and locally uniformly for $z \in \Omega_\rho \setminus L_{1/r}$ and $z \in G_{1/\rho} \setminus L_r$, respectively, and that

$$(41) \quad \sum_{k=0}^\infty f_{n,2k+2}(z) = O(r^{2n}), \quad \sum_{k=0}^\infty f_{n,2k+1}(z) = O(r^n)$$

locally uniformly as $n \rightarrow \infty$ in their respective domains of definition.

From the very definition of $f_{n,2k+2}$ in [3, equation (9)], we see that for $k \geq 0$, $f_{n,2k+2}(\psi(w))$ has an analytic continuation $f_{n,2k+2}^*(w)$ to $\mathbb{C} \setminus \mathbb{T}_{1/r}$ given by

$$(42) \quad f_{n,2k+2}^*(w) = \frac{1}{2\pi i} \oint_{\mathbb{T}_{1/r}} \frac{f_{n,2k+1}(\psi(t)) t^{-n-1} dt}{t - w}, \quad |w| \neq 1/r,$$

and if we define

$$F_n(w) := \sum_{k=0}^\infty f_{n,2k+2}^*(w), \quad |w| \neq 1/r, \quad n \geq 0,$$

then by (41) and (42),

$$(43) \quad F_n(w) = O(r^{2n}) \quad \text{and} \quad F_n'(w) = O(r^{2n})$$

locally uniformly on $|w| \neq 1/r$ as $n \rightarrow \infty$.

According to Theorem 1.1 of [3], for $z \in G_r$ and n large,

$$(44) \quad (n+1)[\phi'(\infty)]^{n+1}P_n(z) = \frac{\varphi'(z)}{2\pi i} \oint_{L_r} \frac{(\sum_{k=0}^{\infty} f_{n,2k}(\xi)) \varphi'(\xi) \phi(\xi)^{n+1} d\xi}{[\varphi(\xi) - \varphi(z)]^2}.$$

Deforming the contour of integration L_r into L_1 does not change the value of this last integral for values of $z \in G_r$, and leaves a function that is analytic in G_1 . By the uniqueness of the analytic continuation and integrating by parts in (44) followed by the change of variables $\xi = \psi(w)$, we obtain

$$(45) \quad \begin{aligned} P_n(z) &= \frac{\varphi'(z)}{(n+1)[\phi'(\infty)]^{n+1}2\pi i} \oint_{\mathbb{T}_1} \frac{[(1 + F_n(w))w^{n+1}]' dw}{\varphi(\psi(w)) - \varphi(z)} \\ &= \frac{\varphi'(z)}{[\phi'(\infty)]^{n+1}2\pi i} \oint_{\mathbb{T}_1} \frac{w^n(1 + K_n^*(w))dw}{\varphi(\psi(w)) - \varphi(z)}, \quad z \in G_1, \end{aligned}$$

with $K_n^*(w) = F_n(w) + wF_n'(w)/(n+1)$, and so multiplying (45) by κ_n , letting

$$K_n(w) := \frac{\kappa_n K_n^*(w)}{\sqrt{n+1}[\phi'(\infty)]^{n+1}} + \frac{\kappa_n}{\sqrt{n+1}[\phi'(\infty)]^{n+1}} - 1,$$

and using (43) and (40), we arrive at Proposition 1.1.

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